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Sequences of Positive Contractions on *AM*-Spaces

H. O. FLÖSSER

Fachbereich Mathematik, Gesamthocschule Paderborn, D-4790 Paderborn, West Germany

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1. INTRODUCTION

A classical theorem of P. P. Korovkin states that a sequence (T_n) of positive linear operators on $C[0, 1]$ converges pointwise to the identity if the three sequences $(T_n(f_i))$ converge to f_i for $f_i(x) = x^i$, $i = 0, 1, 2$. Since its publication in 1953 many authors have generalized this theorem to more general function spaces than $C[0, 1]$.

In the fundamental papers [2, 3], Berens and Lorentz described the way which leads from Korovkin's original theorem to more abstract problems. It goes as follows:

Let E be a Banach lattice, and $\mathscr{L}$ a class of continuous linear operators on E. Given a subset M of E we define the *$\mathscr{L}$-shadow* $\mathscr{L}K(M)$ of M by:

$f \in \mathscr{L}K(M)$ iff for any norm-bounded net $(T_\alpha) \subset \mathscr{L}$ the convergence $\lim_\alpha T_\alpha(p) = p$ for all $p \in M$ implies $\lim_\alpha T_\alpha(f) = f$. (1.1)

The most prominent examples $\mathscr{L}$ considered in the literature are

$$\mathscr{L}^+ \text{: the class of all positive linear operators,} \tag{1.2}$$

$$\mathscr{L}_1 \text{: the class of all linear contractions,} \tag{1.3}$$

$$\mathscr{L}_1^+ = \mathscr{L}^+ \cap \mathscr{L}_1 \text{: the class of all positive linear contractions.} \tag{1.4}$$

The problem, of course, is to determine the $\mathscr{L}$-shadow of M, and to characterize those M whose $\mathscr{L}$-shadows are the whole space E.

In the case $E = C(X)$, X compact metric, and $\mathscr{L} = \mathscr{L}^+$ this problem was solved in [2] using what nowadays is called the *uniqueness closure* of M and upper and lower envelopes. Meanwhile, an exact description of the $\mathscr{L}^+$-shadow was obtained in arbitrary AM-spaces using the same concepts but different methods [8, 13]. In this paper we shall be concerned with the $\mathscr{L}_1^+$-

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shadow of a subset M of an AM-space. Not many results are known in this direction except in the case $E = C(X)$, X compact, in contrast to the situation in AL-spaces where the $\mathscr{L}_1^+$-shadow of a subset M coincides with the closed vector sublattice generated by M([2]; in fact this is true for all L^p-spaces, $1 \leqslant p < \infty$). For M in $C(X)$ in [3] Berens and Lorentz gave necessary and sufficient conditions for $\mathscr{L}_1^+ K(M) = C(X)$ to happen. Other papers in this connection are [10, 12].

We shall give two descriptions of a lower estimate for $\mathscr{L}_1^+ K(M)$, M a subset of the AM-space E. In the case E has a completely regular structure space it is shown that these estimate s are sharp: Two descriptions of $\mathscr{L}_1^+ K(M)$ are obtained. In particular, for $E = C_0(X)$, the Banach of all real valued continuous functions on the locally compact X vanishing at infinity, with sup-norm and pointwise defined order, these descriptions apply. For separable AM-spaces even more is true: Here the $\mathscr{L}_1^+$-shadow of M coincides with the sequentially defined $\mathscr{L}_1^+$-shadow and with the lower estimates given before.

The paper is organized in the following manner. In the rest of this section we explain the notations and definitions used in the sequel. In Section 2 we collect our results and give some examples. Finally, Section 3 is devoted to the proofs of our theorems.

As the $\mathscr{L}_1^+$-shadow always contains the linear subspace of E generated by M we shall assume M to be a subspace and write L instead of M. Furthermore, instead of $\mathscr{L}_1^+ K(L)$ we shall simply write $K_1(L)$, instead of $\mathscr{L}^+ K(L)$ simply $K(L)$. The $\mathscr{L}_1^+$-shadow $K_1(L)$ of L will also be called the *positive contractive shadow* of L, $K(L)$ the *positive shadow*. The *sequential positive contractive shadow* of L, denoted by $K_1^\sigma(L)$ is the set which we obtain when in (1.1) only sequences (T_n) in $\mathscr{L}_1^+$ are allowed.

Whenever E is a Banach lattice (see [11] for notations and basic properties) we denote by $V(E)$ the set of all real valued (continuous) linear lattice homomorphisms on E. Thus

$$V(E) = \{\delta \in E' \mid \forall e, f \in E \qquad \delta(e \wedge f) = \delta(e) \wedge \delta(f)\};$$

let $V(E)_1 = \{\delta \in V(E) \mid \|\delta\| = 1\}$. We define the *contractive uniqueness closure* $E_1(L)$ of a subspace L by

$$E_1(L) = \Big\{ e \in E \mid \forall \delta \in V(E)_1 \ \forall \mu \in E'_+ \quad \delta \underset{L}{=} \mu$$
$$\text{and } \|\mu\| \leqslant 1 \text{ imply } \delta(e) = \mu(e) \Big\}.$$

The *strong contractive uniqueness closure* $\bar{E}_1(L)$ is obtained when in the definition of $E_1(L)$ the set $V(E)_1$ is replaced by its $\sigma(E', E)$-closure.

2. Results and Examples

The letter E will always denote an AM-space, L a subspace of E. The dual E' of E is an AL-space, the bidual E'' an AM-space with order-unit $\mathbb{1}$ and order-unit norm. As a function on $E'_+ := \{\mu \in E' \mid \mu \geqslant 0\}$ the order-unit $\mathbb{1}$ is given by $\mathbb{1}(\mu) = \|\mu\|$; thus it is lower semicontinuous with respect to $\sigma(E', E)$ on E'_+.

We consider E canonically embedded in E''. It is not only a subspace but also a vector sublattice of E''. The set $L + \mathbb{R}_+ \cdot \mathbb{1}$ is a convex cone in E''.

For $A \subset E''$ denote by $\hat{A}$ the set of all finite infima of elements of A, i.e.,

$$\hat{A} = \{\bigwedge A' \mid \phi \neq A' \subset A \text{ finite}\}.$$

Similarly,

$$\check{A} = \{\bigvee A' \mid \phi \neq A' \subset A \text{ finite}\}$$

is the set of all finite suprema of elements of A.

If $A = L + \mathbb{R}_+ \mathbb{1}$ then every element of $\hat{A} = (L + \mathbb{R}_+ \mathbb{1})\hat{}$ is lower semicontinuous with respect to $\sigma(E', E)$ on E'_+. The same is true for the elements in the (norm) closure $(L + \mathbb{R}_+ \mathbb{1})^{\bar{\wedge}}$. Because of $(L - \mathbb{R}_+ \mathbb{1})^{\bar{\vee}} = -(L + \mathbb{R}_+ \mathbb{1})^{\bar{\wedge}}$ every element of $(L - \mathbb{R}_+ \mathbb{1})^{\bar{\vee}}$ is upper semi-continuous on E'_+ with respect to $\sigma(E', E)$. Thus, every $f \in (L + \mathbb{R}_+ \mathbb{1})^{\bar{\wedge}} \cap (L - \mathbb{R}_+ \mathbb{1})^{\bar{\vee}}$ is $\sigma(E', E)$-continuous on E'_+ and can be considered as an element of E.

The following result (as well as its proof) is inspired by [7, Satz 4.2]. It gives an alternative description of the strong contractive uniques closure of L, thus via Theorem 2.9 of the positive contractive shadow of L.

Theorem 2.1. *The strong contractive uniqueness closure $\bar{E}_1(L)$ coincides with $(L + \mathbb{R}_+ \mathbb{1})^{\bar{\wedge}} \cap (L - \mathbb{R}_+ \mathbb{1})^{\bar{\vee}}$. In general it is properly contained in the contractive uniqueness closure $E_1(L)$ of L.*

Examples 2.2. 1. Let $E = C(X)$, X compact, be endowed with the sup-norm. Then $V(E)_1$ is homomorphic to X (via the embedding $\varepsilon : x \to \varepsilon_x$), in particular, it is compact and therefore $\sigma(E', E)$-closed. Thus $\bar{E}_1(L) = E_1(L)$. Furthermore the order-unit $\mathbb{1}$ of E'' is the constant function 1_X on X, hence

$$(L + \mathbb{R}_+ 1_X)^{\bar{\wedge}} \cap (L - \mathbb{R}_+ 1_X)^{\bar{\vee}} = E_1(L) = \bar{E}_1(L).$$

2. Let $E = C_0(X)$, X locally compact, be endorsed with the sup-norm. Denote by $C_b(X)$ the space of all real valued bounded continuous functions on X, by $M_b(X)$ the space of all bounded Radon measures on X and by $M(\beta X)$ the one of all Radon measures on the Stone–Čech-compactification βX of X. Since every $\mu \in M_b(X)$ integrates all functions $f \in C_b(X)$, we may

consider $M_b(X)$ as an ideal in $M(\beta X)$. The duality $\langle C_b(X), M_b(X)\rangle$ is obviously separating; thus the canonical embedding

$$C_b(X) \to E''$$
$$f \to \tilde{f}: \left(\mu \to \int_X f d\mu\right)$$

is a linear lattice isomorphism of $C_b(X)$ into E'' mapping the constant function 1_X on X to the order-unit $\mathbb{1}$ of E''.

We therefore can identify $C_b(X)$ with a vector sublattice of E''. The norm induced by E'' on $C_b(X)$ is the order-unit norm defined by $\mathbb{1} = \tilde{1}_X$; hence it coincides with the sup-norm on $C_b(X)$. Thus

$$(L + \mathbb{R}_+ \mathbb{1})^{\frown} \cap (L - \mathbb{R}_+ \mathbb{1})^{\smile} = (L + \mathbb{R}_+ 1_X)^{\frown} \cap (L - \mathbb{R}_+ 1_X)^{\smile}$$

($\frown$ and $\smile$ on the right-hand side with respect to the pointwise order in $C_b(X)$, closure with respect to the sup-norm) and

$$\bar{E}_1(L) = (L + \mathbb{R}_+ 1_X)^{\frown} \cap (L - \mathbb{R}_+ 1_X)^{\smile}.$$

Now suppose the subspace L satisfies the following condition (P):

(P) For every $x \in X$ there is a positive $f \in L$ such that $f(x) \neq 0$.

Then $E_1(L) = \bar{E}_1(L)$. Indeed, for $E = C_0(X)$ we know $V(E)_1 = \{\varepsilon_x \mid x \in X\}$ and $\overline{V(E)_1} = V(E)_1 \cup \{0\}$. Therefore $E_1(L) = \bar{E}_1(L)$ whenever $\mu \in E'_+$, $\mu =_L 0$ implies $\mu = 0$.

But $\mu \in E'_+$ is a positive bounded Radon measure on X. It is zero, if $\mu(K) = 0$ for all compact $K \subset X$. Thus let $K \subset X$ be compact. Via a simple compactness argument condition (P) yields a positive $f_K \in L$ such that $f_K(x) \geqslant 1$ for all $x \in K$. Then $0 \leqslant \mu(K) = \int_K d\mu \leqslant \int_K f_K\, d\mu \leqslant \int_X f_K\, d\mu = 0$ if $\mu =_L 0$.

3. As in Example 2 let $E = C_0(X)$, but suppose the subspace L satisfies

(PP) For all $(x, y) \in X^2$, $x \neq y$, there is a positive $g \in L + \mathbb{R}_+$ such that $g(x) \neq 0$ and $g(y) = 0$.

Then $E_1(L)$ is the closed ideal generated by L in $C_0(X)$. To prove this assume we have a positive bounded Radon measure μ on X with norm $\leqslant 1$ and a point $y \in X$ such that $\int_X f d\mu = f(y)$ for all $f \in L$. Let K be a compact subset of $X \setminus \{y\}$. Compactness of K and condition (PP) applied to (x, y), $x \in K$, yield a function $f \in L$ and a real number $r \geqslant 0$ such that

$$1 \leqslant f(x) + r \quad \text{for all } x \in K, \qquad 0 = f(y) + r \quad \text{and} \quad 0 \leqslant f + r.$$

From this it follows

$$0 \leqslant \mu(K) = \int_K d\mu \leqslant \int_K (f+r)\, d\mu = \int_X f\, d\mu + r \int_X d\mu$$
$$\leqslant f(y) + r\,\|\mu\| \leqslant f(y) + r = 0.$$

We conclude $\mu(K) = 0$ and supp $\mu = \{y\}$ as $K \subset X \setminus \{y\}$ was an arbitrary compact set. This yields $\mu = r\varepsilon_y$ for some $0 \leqslant r \leqslant 1$ and $f(y) = rf(y)$ for all $f \in L$. Thus, either

(a) $\mu = r\varepsilon_y$ for some $0 \leqslant r < 1$ and $\varepsilon_y =_L 0$, or

(b) $\mu = \varepsilon_y$.

In case (a) $\varepsilon_y = 0$ on the closed ideal J generated by L and $\mu_J = 0$; in both cases $\mu_J = \varepsilon_y$ and therefore $J \subset E_1(L)$.

The converse inclusion comes from

$$J = \bigcap \left\{ \varepsilon_x^{-1}(0) \mid \varepsilon_x \underset{L}{=} 0 \right\}.$$

4. Let $E = c_0 := C_0(\mathbb{N})$, L the kernel of the functional μ given by the sequence (2^{-n}). Then

(a) L satisfies (PP), but not (P);

(b) $\bar{E}_1(L) = L$, but $E_1(L) = c_0$.

Indeed, since the functional μ is strictly positive, there is no $0 \leqslant f \in L$ with $f \neq 0$; thus (P) is not satisfied.

Now, given $(k, l) \in \mathbb{N}^2$, $k \neq l$, define $f\colon \mathbb{N} \to \mathbb{R}$ by

$$\begin{aligned} f(n) &= 0 && \text{if} \quad n \neq l, l+1 \\ &= -1 && \text{if} \quad n = l \\ &= 2 && \text{if} \quad n = l+1. \end{aligned}$$

Thus $f \in L$ because of $\mu(f) = \sum_{n \in \mathbb{N}} 2^{-n} f(n) = -2^{-l} + 2 \cdot 2^{-(l+1)} = 0$; furthermore $g := f + 1_{\mathbb{N}} \geqslant 0$ and $g(n) = 0$ iff $n = l$. Thus $g(l) = 0$, $g(k) \neq 0$, and (PP) holds true.

By the foregoing example $E_1(L) = c_0$ (the ideal generated by the maximal subspace L is c_0).

Now $0 \in \overline{V(E)_1} = \overline{\{\varepsilon_n \mid n \in \mathbb{N}\}}$ and $\mu =_L 0$ imply $\mu = 0$ on $\bar{E}_1(L)$, i.e., $\bar{E}_1(L) \subset L$.

The connection of the results obtained so far with Korovkin approximation becomes clear by the next theorem:

THEOREM 2.3. *Besides E let F be an AM-space, too. Suppose* (T_α) *is a net of positive linear contractions from E into F, S a linear lattice homomorphism from E into F satisfying* $S'(\delta) \in \overline{V(E)}_1$ *for all* $\delta \in V(F)_1$, *such that* $\lim_\alpha T_\alpha(f)$, *for all* $f \in L$. *Then* $\lim_\alpha T_\alpha(f) = S(f)$ *for all* $f \in \bar{E}_1(L)$.

Taking $F = E$, S the identity on E, we obtain

COROLLARY 2.4. *We always have*

$$(L + \mathbb{R}_+ \mathbb{1})^{\curlywedge} \cap (L - \mathbb{R}_+ \mathbb{1})^{\curlyvee} = \bar{E}_1(L) \subset K(L)_1.$$

From Examples 2.2.2 and 2.2.3 we may extract the following peak-point criterion for $K(L)_1 = E$:

COROLLARY 2.5. *Let* $E = C_0(X)$ *and L a subspace satisfying*

($\overline{\text{P}}$) *L contains a strictly positive function*;

($\overline{\text{PP}}$) *for every* $x \in X$ *there is an* $f \in L$ *such that* $f(x) = 1$ *but* $f(y) < 1$ *for all* $y \in X$, $y \neq x$.

Then $K_1(L) = C_0(X)$. If X is compact, ($\overline{\text{P}}$) may be omitted.

EXAMPLES 2.6. 1. Let $E = C(X)$, X compact. By Example 2.2.1 $E_1(L) = \bar{E}_1(L)$, by 2.4 $E_1(L) \subset K(L)_1$. This gives the sufficiency part of Theorem 3 in [3]. The condition ($\overline{\text{PP}}$) is nothing else than the peak-point condition (P_1^+) in 1.5 of [3].

2. Let $E = \{f \in C[0, 1] \mid f(0) = 0\}$, L the subspace spanned by the two functions $f_1 : x \to x$ and $f_2 : x \to x^2$. Then $K_1(L) = E$.

Indeed, we can identify E with $C_0(X)$ for $X = (0, 1]$. The function f_1 is strictly positive on X, therefore ($\overline{\text{P}}$) is satisfied. To verify ($\overline{\text{PP}}$) for given $x \in X$ define f by $f = (1/x^2) f_2 - (3/x) f_1$, i.e., $f(y) = 1 - (y/x - 1)^2$.

3. (See [1, Beispiel 4].) Let $E = C[0, \infty)$, L the subspace spanned by the two functions $f_i : x \to e^{-t_i x}$ where $0 < t_1 < t_2$. Condition ($\overline{\text{P}}$) holds true (take f_1). Now, given $x \in [0, \infty)$ let

$$f(y) = \frac{t_2}{t_2 - t_1} e^{t_1(x-y)} - \frac{t_1}{t_2 - t_1} e^{t_2(x-y)};$$

the function f has its only maximum point at $y = x$ (differentiate to verify) where $f(x) = 1$. Thus ($\overline{\text{PP}}$) is veryfied, too, and by Corollary 2.5 $K_1(L) = C_0[0, \infty)$.

For the moment consider the problem of characterizing the shadow of L

in E without our "contractive assumptions". It is known that in AM-spaces the inclusion $E(L) \subset K(L)$ is always true (see, for example, [8]), where

$$E(L) = \{f \in E \mid \forall \delta \in V(E) \, \forall \mu \in E'_+ \, \delta \underset{L}{=} \mu \Rightarrow \delta(f) = \mu(f)\}.$$

The corresponding "contractive result" which one might have expected, namely $E_1(L) \subset K_1(L)$, is false in general (see Theorem 2.9). But as in [7], where we also had to use the strong uniqueness closure instead of $E_1(L)$, we still have pointwise convergence on $E_1(L)$. To be more precise following theorem is true:

THEOREM 2.7. *Let E and F be arbitrary Banach lattices, $L \subset E$ a subspace. Suppose (T_α) is net of positive contractions from E into F, S a linear lattice homomorphism from E into F satisfying $\|S'(\delta)\| = \|\delta\|$ for all $\delta \in V(F)$. Then from*

(i) $\forall \delta \in V(F) \, \forall f \in L \, \lim_\alpha \delta(T_\alpha(f)) = \delta(S(f))$ *follows*

(ii) $\forall \delta \in V(F) \, \forall f \in E_1(L) \, \lim_\alpha \delta(T_\alpha(f)) = \delta(S(f))$.

Observe that for $E = F = C_0(X)$ the convergence expressed in (ii) of Theorem 2.7 really is the pointwise convergence. Theorem 2.7 serves well to determine the stationary contractive shadow of a subspace.

COROLLARY 2.8. *Let E, F be Banach lattices such that $V(F)$ separates the points of F, T a positive linear contraction from E into F, S a linear lattice homomorphism from E into F satisfying $\|S'(\delta)\| = \|\delta\|$ for all $\delta \in V(F)$. If T and S coincide on the subspace L of E then they coincide on the uniqueness closure $E_1(L)$ of L.*

For arbitrary AM-spaces it is well known [13] that the uniqueness closure $E(L)$ of a subspace L is not only contained in the shadow of L but actually coincides with it. Whether the corresponding statement $\bar{E}_1(L) = K_1(L)$ is true or not in arbitrary AM-spaces we hve to leave as an open problem. (The arguments to prove $K(L) \subset E(L)$ seem not to apply to our case due to the fact that we have to use $\bar{E}_1(L)$ instead of $E_1(L)$). But, similar as in [8], for a large class of AM-spaces we are able to prove $K_1(L) \subset \bar{E}_1(L)$. This class comprises all $C_0(X)$ and all AM-spaces containing a topological order-unit (i.e., an element $0 \leqslant u$ which as a functional on E' is strictly positive on $E'_+ \setminus \{0\}$), in particular all separable AM-spaces.

The idea of the proof is very old and has been used by quite a number of authors at similar occasions. (See [2, Theorem 2], for example.) To be able to use this idea in the context of AM-spaces we have to restrict ourselves to those which have a nice enough structure space.

The structure space Max (E) of an AM-space E was introduced by E. G.

Effros and studied in very great detail by A. Goullet de Rugy in [9]. It is the set $V(E)_1$ endowed with the facial topology the closed sets of which are the traces on $V(E)_1$ of the $\sigma(E', E)$-closed faces of E'_+.

By an example due to Goullet de Rugy it is known that there are AM-spaces the structure space of which is trivial in the sense that there are no real valued continuous functions on it besides the constants [9, last remark]. But for $E = C_0(X)$ the structure space is homeomorphic to X, for E containing a topological order unit u it is homeomorphic to $X_u = \{\delta \in V(E) \mid \delta(u) = 1\}$ with $\sigma(E', E)$ induced on X_u.

THEOREM 2.9. *Suppose the structure space of the AM-space E is completely regular. Then*

$$(L + \mathbb{R}_+ \mathbb{1})^{\curlyvee} \cap (L - \mathbb{R}_+ \mathbb{1})^{\curlywedge} = \bar{E}_1(L) = K_1(L).$$

COROLLARY 2.10. *Let $E = C_0(X)$, X locally compact. For every subspace L of E we have*

$$(L + \mathbb{R}_+ 1_X)^{\curlyvee} \cap (L - \mathbb{R}_+ 1_X)^{\curlywedge} = \bar{E}_1(L) = K_1(L).$$

If the AM-space E is separable its structure space is completely regular [9, 2.7 and 2.11], thus Theorem 2.9 applies to this case. But, indeed, one can prove more, namely, one can determine the sequential positive contractive shadow of L:

THEOREM 2.11. *Let E be separable AM-space. Then for every subspace L of E we have*

$$(L + \mathbb{R}_+ 1)^{\curlyvee} \cap (L - \mathbb{R}_+ 1)^{\curlywedge} = \bar{E}_1(L) = K_1(L) = K_1^{\sigma}(L).$$

EXAMPLE 2.12. 1. Let $E = C(X)$, X compact. By Example 2.2.1 and Theorem 2.9 $E_1(L) = K_1(L)$. If X is metrizable even $E_1(L) = K_1^{\sigma}(L)$. This gives Theorem 3 of [3].

2. Let $E = C[0, 1]$, L the subspace spanned by the two functions $f_i(x) = x^i$, $i = 1, 2$; to prove $K_1^{\sigma}(L) = \{f \in C[0, 1] \mid f(0) = 0\}$ we first check condition (PP) of Example 2.2.3. Let $(x, y) \in [0, 1]^2$, $x \neq y$.

(a) $x = 0$, $y \neq 0$. Define $g(t) = (t - y)^2$; then $0 \leqslant g \in L + \mathbb{R}_+$, $g(0) = y^2 \neq 0$ and $g(y) = 0$.

(b) $x \neq 0$; $y = 0$. Let $g(t) = t$; then $0 \leqslant g \in L \subset L + \mathbb{R}_+$, $g(x) = x^2 \neq 0$, $g(y) = g(0) = 0$.

(c) $0 \neq x, y$: Proceed as in Example 2.6.2.

Thus, $E_1(L)$ is equal to the closed ideal generated by f_1 and f_2, i.e.,

$E_1(L) = \{f \in C[0, 1] \mid f(0) = 0\}$. On the other hand $E_1(L) = K_1^\sigma(L)$ by 2.12.1.

3. Once more consider the Example 2.2.4. There we found $\bar{E}_1(L) = L$, but $E_1(L) = c_0$. By Theorem 2.11 $K_1^\sigma(L) = L$. But for any sequence of operators (T_n) admitted in the definition of $K_1^\sigma(L)$ and for any $f \in c_0$ the sequence $(T_n f)$ still converges pointwise on $\mathbb{N}$ to f. In particular, any positive contraction T from c_0 to c_0 coinciding on L with the identity is the identity (Corollary 2.8) in contrast to the situation with non-contractive assumption. By [8, Proof of Theorem 5.6], $K(L)$ (in c_0!) is equal to the stationary shadow of L; this means for any $f \notin L$ there is a positive operator $T: c_0 \to c_0$ such that $T =_L \mathrm{Id}$, but $Tf \neq f$.

Once additional remark: The space c_0 shares the property that every closed vector sublatice is the range of a positive contractive projection with the $L^p(\mu)$, $1 \leqslant p < \infty$. But in contrast to the case $L^p(\mu)$ $K_1(L)$ in c_0 is in general strictly contained in the closed vector sublattice generated by L as seen by the above L.

This, of course, is due to the lack of monotonicity of the norm on c_0.

3. PROOFS

We start with a proposition which serves as a lemma in the proof of Theorem 2.1 but which bears some interest in itself. To make life easier we first recall some well known facts on Banach lattices, particularly on AM-spaces.

Let E be Banach lattice, K_E the positive part of its dual unit ball: $K_E = \{\mu \in E' \mid \mu \geqslant 0, \|\mu\| \leqslant 1\}$. Equip K_E with the weak*-topology. Then we may identify E with $A_0(K_E) = \{f \in A(K_E) \mid f(0) = 0\}$, where $A(K_E)$ denotes the space of all real valued continuous affine functions on K_E. The state space $\operatorname{st} A(K_E)$ is defined by

$$\operatorname{st} A(K_E) := \{\mu \in A(K_E)' \mid \mu \geqslant 0, \mu(1) = 1\}.$$

Embedding K_E into $\operatorname{st} A(K_E)$ (by point evaluation) yields a 1-1 correspondence between the extreme points $\operatorname{ex} K_E$ of K_E and the extreme points of st $A(K_E)$.

The set $V(E)$ of all realvalued linear lattice homomorphisms on E coincides with the set of functionals which 1y on some extremal ray of the cone $E'_+ = \{\mu \in E' \mid \mu \geqslant 0\}$ Thus $V(E)_1 = \{\delta \in V(E) \mid \|\delta\| = 1\} \subset \operatorname{ex} K_E$. The space E is an AM-space iff $\operatorname{ex} K_E = V(E)_1 \cup \{0\}$. In this case K_E is a cap of E'_+ and a simplex.

Now consider E and its bidual E'', K_E and $K_{E''}$ the respective positive parts of their dual unit balls. Then $A(K_{E''}) = A(K_E)''$ because of

$$A(K_{E''}) = A_0(K_{E''}) + \mathbb{R} = E'' + \mathbb{R} = (E + \mathbb{R})'' = (A_0(K_E) + \mathbb{R})'' = A(K_E)''. \tag{3.1}$$

Finally from [7] we recall Lemma 4.1: If ρ: $A(K_E)'' \to A(K_E)'$ denotes the restriction map we have

$$\rho(\text{ex st}\, A(K_E)'') = \overline{\text{ex st}\, A(K_E)} \tag{3.2}$$

(weak*-closure), whenever K_E is a simplex.

Proposition 3.1. *Consider the AM-space E to be canonically embedded in its bidual E″ and denote by R the restriction map from E‴ onto E′. Then* $R\ (V(E'')_1) = \overline{V(E)_1}$.

Proof. Let $\delta \in V(E'')_1$; then $0 \neq \delta \in \text{ex}\, K_{E''}$. Thus the evaluation ε_δ at δ is an extreme point of $\text{st}\, A(K_{E''})$, by (3.1) one of $\text{st}\, A(K_E)''$. By (3.2) there is a net $(\tilde{\delta}_\alpha)$ in ex st $A(K_E)$ weak*-converging to the restriction $\rho(\varepsilon_\delta)$ of ε_δ on $A(K_E)$. But every $\tilde{\delta}_\alpha$ is the point evaluation at some $\delta_\alpha \in \text{ex}\, K_E = V(E)_1 \cup \{0\}$. Thus for any $e \in E$ we have

$$\delta(e) = \rho(\varepsilon_\delta)(e) = \lim\, \tilde{\delta}_\alpha(e) = \lim\, \delta_\alpha(e)$$

and $R(\delta) \in \overline{V(E)_1 \cup \{0\}} = \overline{V(E)_1} \cup \{0\}$. For the inclusion $R(V(E'')_1) \subset \overline{V(E)_1}$ it remains to be shown that $R(\delta) \neq 0$ for $0 \notin \overline{V(E)_1}$.

Now $\overline{V(E)_1}$ is compact as a closed subset of K_E. A simple compactness argument yields $0 \leqslant u \in E$ which evaluated on $\overline{V(E)_1}$ is strictly positive whenever $0 \notin \overline{V(E)_1}$. As every element $0 \leqslant f$ of E attains its norm on $\overline{V(E)_1}$, u is an order unit in E. Furthermore, the given norm and the order-unit norm of u are equivalent (with respect to both norms E is complete). It follows that u is an order-unit in E'', too. Thus, $R(\delta)\ (u) \neq 0$ for $\delta \neq 0$, in particular for $\delta \in V(E'')_1$.

To prove the converse inclusion we recall that $V(E)_1 \subset R(V(E'')_1)$ by an extension theorem for extremal functionals proved in [5] (Corollary 2.5). Since E'' is an AM-space with order-unit $\mathbb{1}$ $V(E'')_1 = \{\delta \in V(E'') \mid \delta(\mathbb{1}) = 1\}$ is weak*-compact. As R is continuous with respect to the respective weak*-topologies $R(V(E'')_1)$ is weak*-compact, too. Therefore $\overline{V(E)_1} \subset R(V(E'')_1)$.

If E is an AM-space the unit ball B_E of its dual can be described as $B_E = \text{co}\,(K_E \cup -K_E)$. The set of extreme points of B_E is given by $\text{ex}\, B_E = \text{ex}\, K_E \cup -\text{ex}\, K_E$. This observation yields the following corollary.

Corollary 3.2. *Let E be an AM-space, E″ its bidual and* B_E *and* $B_{E''}$

the unit balls of the respective duals. The restriction map $R\colon E''' \to E'$ *maps* $\operatorname{ex} B_{E''}$ *onto* $\overline{\operatorname{ex} B_{E'}}$.

Now we are ready to prove Theorem 2.1:

Proof of Theorem 2.1. Suppose $\pm e \in (L + \mathbb{R}_+ \mathbb{1})^{\frown}$; to prove $e \in \overline{E_1}(L)$ let $\delta \in \overline{V(E)_1}$, $\mu \in E'_+$ with $\|\mu\| \leqslant 1$ and $\delta =_L \mu$ be given. By Proposition 3.1 there is $\tilde{\delta} \in V(E'')_1$ such that $R(\tilde{\delta}) = \delta$; there is also a normequal positive extension $\tilde{\mu}$ of μ on E''. Because of $\tilde{\delta} =_L \tilde{\mu}$ and $\tilde{\mu}(\mathbb{1}) = \|\tilde{\mu}\| \leqslant 1 = \|\tilde{\delta}\| = \tilde{\delta}(\mathbb{1})$ we obtain $\tilde{\mu}(f) \leqslant \tilde{\delta}(f)$ for all $f \in L + \mathbb{R}_+ \mathbb{1}$. Now let $f = \bigwedge_{i=1}^n f_i$, $f_i \in L + \mathbb{R}_+ \mathbb{1}$. Then $\tilde{\mu}(f) \leqslant \bigwedge_{i=1}^n \tilde{\mu}(f_i) \leqslant \bigwedge_{i=1}^n \tilde{\delta}(f_i) = \tilde{\delta}(f)$. Finally, because of the continuity of $\tilde{\mu}$ and $\tilde{\delta}$ we also have $\tilde{\mu}(f) \leqslant \tilde{\delta}(f)$ for all $f \in (L + \mathbb{R}_+ \mathbb{1})^{\frown}$. Our assumption on e yields $\tilde{\mu}(\pm e) \leqslant \tilde{\delta}(\pm e)$ and $\tilde{\mu}(e) = \tilde{\delta}(e)$. By remarks at the beginning of Section 2 $e \in E$, whence $\mu(e) = \delta(e)$ and $e \in \bar{E}_1(L)$.

Now assume the proven inclusion is strict. Then there exists $e \in \bar{E}_1(L)$ which does not belong to $(L + \mathbb{R}_+ \mathbb{1})^{\frown}$. Using a theorem of Choquet and Deny ([4]; this theorem was rediscovered in connection with Korovkin approximation and generalized to arbitrary *AM*-spaces in [8]) we find $\delta \in V(E'')$ and $\mu \in E'''_+$ such that $\mu(f) \leqslant \delta(f)$ for all $f \in L + \mathbb{R}_+ \mathbb{1}$ but $\delta(e) < \mu(e)$. Since $\delta = 0$ implies $\|\mu\| = \mu(\mathbb{1}) \leqslant \delta(\mathbb{1}) = 0$ we may suppose $\|\delta\| = 1$. Then by Proposition 3.1 $R(\delta) \in \overline{V(E)_1}$ and, of course, $R(\mu) \in E'_+$ with $[\![R(\mu)]\!] \leqslant [\![\mu]\!] = \mu(\mathbb{1}) \leqslant \delta(\mathbb{1}) = 1$. As L is a vector space the inequality $R(\mu) \leqslant R(\delta)$ on L implies $R(\mu) = R(\delta)$ on L. But then $\mu(e) = R(\mu)\ (e) = \delta(e)$ because of $e \in \bar{E}_1(L)$ which contradicts $\delta(e) < \mu(e)$.

Finally the inclusion $\bar{E}_1(L) \subset E_1(L)$ is trivial, while Example 2.2.4 shows that in general it is strict.

Proof of Theorem 2.3. Assume the statement of Theorem 2.3 is false. Then there exists a net (T_α) of linear positive contractions from E into F, a linear lattice homomorphism S from E into F such that

$$S'(\delta) \in \overline{V(E)_1} \qquad \text{for all} \quad \delta \in V(F)_1 \tag{3.3}$$

$$\lim_\alpha T_\alpha(f) = S(f) \qquad \text{for all} \quad f \in L$$

but not $\lim_\alpha T_\alpha(e) = S(e)$ for some $e \in \bar{E}_1(L)$. This implies the existence of an $\varepsilon > 0$, a subnet (T_β) of (T_α) and a net $(\delta_\beta) \subset V(F)_1$ such that

$$|\delta_\beta(T_\beta(e) - S(e))| > \varepsilon \qquad \text{for all} \quad \beta. \tag{3.5}$$

(Here we used that the norm convergence in F is the same as the uniform convergence on $V(F)_1$.)

Now because of $\| T'_\beta(\delta_\beta)\| \leqslant \|\delta_\beta\| \leqslant 1$ and $S'(\delta_\beta) \in \overline{V(E)_1}$ there are subnets

(T_γ) of (T_β) and (δ_γ) of (δ_β) such that $(T'_\gamma(\delta_\gamma))$ and $(S'(\delta_\gamma))$ weak*-convergence to μ and δ, respectively. Then $\delta \in \overline{V(E)}_1$ and $\mu \in E'_+$, $\|\mu\| \leqslant 1$.

We also have $\delta =_L \mu$ because of (3.4). Namely,

$$\delta(f) = \lim S'(\delta_\gamma)(f) = \lim \delta_\gamma(S(f)) = \lim \delta_\gamma(T_\gamma(f)) = \mu(f)$$

for all $f \in L$. The assumption on e implies $\mu(e) = \delta(e)$ which by as a similar argument as the last one contradicts (3.5).

The proof of Theorem 2.7 is quite similar to the forgoing and will be sketched only:

Proof of Theorem 2.7. Again assuming that the claim is false, we find (T_α), S such that

$$\|S'(\delta)\| = \|\delta\| \qquad \text{for all} \quad \delta \in V(F), \tag{3.3'}$$

$$\lim_\alpha T_\alpha(f) = S(f) \qquad \text{pointwise on } V(F) \text{ for all } f \in L \tag{3.4'}$$

and, for some $e \in E_1(L)$, some $\delta \in V(F)_1$, some $\varepsilon > 0$ and some subnet (T_β) of (T_α)

$$|\delta(T_\beta(e) - S(e)| > \varepsilon \qquad \text{for all} \quad \beta. \tag{3.5'}$$

Again let μ be the weak*-limit of a subnet $(T'_\gamma(\delta))$ of $(T'_\beta(\delta))$. Then (3.4′) implies $\mu =_L S'(\delta)$ which in turn yields $\mu(e) = S'(\delta)(e)$ (use $S'(\delta) \in V(E)$ and (3.3′)). Again this contradicts (3.5′).

Before proving Theorem 2.9 we recall some facts on the structure space Max(E) of an AM-space E and its connection to the ideal center $Z(E)$ of E. As general reference concerning Max(E) we quote [9], but observe that the center of an AM-space defined there, Definition 2.26, is not the ideal center.

The structure space Max(E) is defined to be the set $V(E)_1 = \{\delta \in V(E) \mid \|\delta\| = 1\}$ $(= \text{ex}\, K_E \backslash \{0\})$ endowed with the facial topology (which is coarser than the topology induced by $\sigma(E', E)$) [9, Definition 1.29]. Another description of Max(E) is the following: Let $V(E)_* = V(E) \backslash \{0\}$ and denote by Str(E) the quotient of $V(E)_*$ over the equivalence relation $\sim$, where $\delta_1 \sim \delta_2$ if there exists $r > 0$ such that $\delta_1 = r\delta_2$. The set Str(E) with the quotient topology of $\sigma(E', E)$ on $V(E)_*$ is, by [9, 1.34], homeomorphic to Max(E). If π is the quotient map $\pi\colon V(E)_* \to \text{Str}(E)$ then the homeomorphism is given by $\pi_{|\text{Max}(E)}$.

Yet another description of Max(E) is the set $\mathscr{M}$ of all closed maximal ideals in E which with the hull-kernel topology is homeomorphic to Str(E).

The ideal center $Z(E)$ is defined as the set of all linear operators on E which are bounded by a multiple of the identity I on E, i.e.,

$$Z(E) = \{T \in \mathscr{L}(E) \mid \exists r \geqslant 0 \quad |T| \leqslant rI\}.$$

By [6, 2.3], $Z(E)$—as an AM-space with order-unit I—is isomorphic to the AM-space $C_b(\mathscr{M})$ of bounded continuous real functions on $\mathscr{M}$. A function φ and the corresponding operator $R_\varphi \in Z(E)$ satisfy

$$\delta(R_\varphi(e)) = \varphi(\delta)\, \delta(e) \qquad \text{for all} \quad \delta \in V(E),\, e \in E, \tag{3.6}$$

(we write $\varphi(\delta)$ for $\varphi(\delta^{-1}(0))$).

The hypothesis on the complete regularity of $\operatorname{Max}(E)$ made in Theorem 2.9 enables us to work with $Z(E)$: We have enough functions in $C_b(\mathscr{M})$, thus enough operators in $Z(E)$ to prove $K_1(L) \subset \bar{E}_1(L)$. Nevertheless the proof is much more complicated than the one for $K(L) \subset E(L)$ given in [8]. This is due to the following facts:

(i) here we don't assume the existence of a topological order unit or at least a topological orthogonal system as we did in [8];

(ii) here the operators constructed in the proof have to have norm $\leqslant 1$; in [8] the net had to be equicontinuous only;

(iii) last but not least we have to deal with $\bar{E}_1(L)$ instead of $E_1(L)$.

Proof of Theorem 2.9. Suppose e does not belong to $\bar{E}_1(L)$; then there exist $\delta_0 \in \overline{V(E)_1}$ and $\mu_0 \in E'_+$ with $\|\mu_0\| \leqslant 1$ such that $\delta_0 =_L \mu_0$ but $\delta_0(e) \neq \mu_0(e)$. We have to prove $e \notin K_1(L)$, i.e., we have to find a net (T_α) of linear positive contractions on E converging pointwise on L to the identity such that $(T_\alpha(e))$ does not converge to e. To do so we consider the two cases (I) $\delta_0 = 0$ and (II) $\delta_0 \neq 0$.

Case (I). $\delta_0 = 0$.

Let $\mathscr{K}$ be a fundamental system of compact subsets of $\operatorname{Str}(E)$, directed upward by inclusion. For any $K \in \mathscr{K}$ choose $\delta_K \in V(E)_1$ such that $\pi(\delta_K) \notin K$ (this is possible because otherwise $\operatorname{Str}(E)$ and $\operatorname{Max}(E)$ were compact; then E had a order unit u, [9, 3.25]. The equivalence of the order unit norm and the given norm would imply $0 \notin \overline{V(E)_1}$ which contradicts the assumption $\delta_0 = 0 \in \overline{V(E)_1}$.) For every $K \in \mathscr{K}$ the complete regularity of $\operatorname{Str}(E)$ assures the existence of a function $\varphi_K \in C_b$ ($\operatorname{Str}(E)$) such that

$$0 \leqslant \varphi_K \leqslant 1, \qquad \varphi_K(\pi(\delta_K)) = 1, \qquad \varphi_K(K) = \{0\}. \tag{3.7}$$

Let R_K be the operator in $Z(E)$ corresponding to φ_K. Then the isomorphism of $C_b(\operatorname{Str}(E))$ and $Z(E)$ implies

$$0 \leqslant R_K \leqslant I \qquad \text{for all } K \in \mathscr{K}. \tag{3.8}$$

Finally, let $A = \{f \in E \mid 0 \leqslant f,\ \|f\| \leqslant 1\}$; as E is an AM-space A is upward directed. Thus, $\mathscr{K} \times A$ with the componentwise defined order is upward directed, too. Now, for every $K \in \mathscr{K}$ and every $f \in A$ define an operator $T_{K,f}$ by

$$T_{K,f}(g) = \mu_0(g)\, R_K(f) + (I - R_K)\ (g), \qquad g \in E.$$

We shall show that $(T_{K,f})$ is a net of linear positive contractions on E such that $\lim T_{K,f}(g) = g$ for all $g \in L$, but not $\lim T_{K,f}(e) = e$.

Linearity of $T_{K,f}$ is clear. Positivity follows by $\mu_0 \geqslant 0$, $R_K \geqslant 0$, $I - R_K \geqslant 0$ (by (3.8)) and $f \geqslant 0$. To prove $\|T_{K,f}\| \leqslant 1$ take $0 \leqslant g \in E$, $[\![g]\!] \leqslant 1$ and $\delta \in V(E)_1$. Using (3.6) we obtain

$$\begin{aligned} 0 \leqslant \delta(T_{K,f}(g)) &= \mu_0(g)\, \varphi_K(\pi(\delta))\, \delta(f) + (1 - \varphi_K(\pi(\delta)))\, \delta(g) \\ &\leqslant \varphi_K(\pi(\delta))\, \mu_0(g) + (1 - \varphi_K(\pi(\delta)))\, \delta(g); \end{aligned}$$

since $0 \leqslant \varphi_K \leqslant 1$ the right-hand side of this inequality is a convex combination of $\mu_0(g)$ and $\delta(g)$. Thus

$$0 \leqslant \delta(T_{K,f}(g)) \leqslant \max\ (\mu_0(g), \delta(g)) \leqslant \max([\![\mu_0]\!]\, [\![\delta]\!])\ \|g\| \leqslant 1$$

and

$$\begin{aligned} \|T_{K,f}\| &= \sup\{\|T_{K,f}(g)\| \mid 0 \leqslant g \in E, [\![g]\!] \leqslant 1\} \\ &= \sup\{\sup\{\delta(T_{K,f}(g)) \mid \delta \in V(E)_1\} \mid 0 \leqslant g \in E, [\![g]\!] \leqslant 1\} \\ &\leqslant 1. \end{aligned}$$

Now fix $g \in L$. Then $\mu_0(g) = \delta_0(g) = 0$ and $T_{K,f}(g)$ becomes simply $T_{K,f}(g) = (I - R_K)(g)$.

To prove $\lim T_{K,f}(g) = g$ it thus suffices to prove $\lim R_K(g) = 0$. Let $\varepsilon > 0$ be arbitrary and $K_\epsilon := \{\delta \in \overline{V(E)_1} \mid |\delta(g)| \geqslant \varepsilon\}$. The set K_ϵ is a $\sigma(E', E)$-compact subset of $V(E)_*$. Therefore $\pi(K_\epsilon)$ is compact in $\mathrm{Str}(E)$ and there exists $K_0 \in \mathscr{K}$ containing $\pi(K_\epsilon)$. Fix an arbitrary $f_0 \in A$. Then for any $K \in \mathscr{K}$, $K \supset K_0$, $f \in A$ with $f \geqslant f_0$ and $\delta \in V(E)_1$ using (3.7) one gets

$$\begin{aligned} |\delta(R_K(g))| &= \varphi_K(\pi(\delta))\ |\delta(g)| \\ &= \begin{cases} 0 & \text{if } \delta \in K_\epsilon \text{ and } \pi(\delta) \in \pi(K_\epsilon) \subset K_0 \subset K \\ < \varepsilon & \text{if } \delta \notin K_\epsilon \text{ and } \varphi_K(\pi(\delta)) \leqslant 1,\ |\delta(g)| < \varepsilon \end{cases}. \end{aligned}$$

Thus $[\![R_K(g)]\!] \leqslant \varepsilon$ for all $K \in \mathscr{K}$ such that $K \supset K_0$. To show that $(T_{K,f}(e))$ does not converge to e we first show

$$\lim \delta_K = 0 \qquad (\text{in } \sigma(E', E)). \tag{3.9}$$

Let $g \in E$, $\varepsilon > 0$ and put $K_\epsilon = \{\delta \in \overline{V(E)_1} \mid |\delta(g)| \geqslant \varepsilon\}$. Again $\pi(K_\epsilon)$ is compact and there is $K_0 \in \mathscr{K}$ such that $K_0 \supset \pi(K_\epsilon)$. Since by choice of δ_K the functional δ_K does not belong to K_ϵ, for $K \supset K_0$ we have $|\delta_K(g)| < \varepsilon$ for all $K \supset K_0$.

Finally assume lim $T_{K,f}(e) = e$. Because of

$$|\delta_K(T_{K,f}(e))| \leqslant |\delta_K(T_{K,f}(e) - e)| + |\delta_K(e)|$$
$$\leqslant \|T_{K,f}(e) - e\| + |\delta_K(e)|$$

this assumption together with (3.9) enforces $(\delta_K(T_{K,f}(e)))$ to converge to 0. But evaluating $\delta_K(T_{K,f}(e))$ we obtain (use (3.6) and (3.7)) $\delta_K(T_{K,f}(e)) = \mu_0(e)\, \delta_K(f)$ which not converges to 0. Indeed, for given $K_0 \in \mathscr{K}, f_0 \in A$ there are $\mathscr{K} \ni K \supset K_0$, $A \ni f \geqslant f_0$ such that

$$|\delta_K(T_{K,f}(e))| > \tfrac{3}{4}\,|\mu_0(e)|:$$

Choose $f_1 \in A$ such that $\delta_{K_0}(f_1) > \frac{3}{4}$, let $f_2 = f_0 \wedge f_1 \in A$, $K = K_0$ and $f = f_2$. Then $|\delta_K(T_{K,f}(e))| = |\mu_0(e)|\, \delta_{K_0}(f_2) > \frac{3}{4}\,|\mu_0(e)|$.

Case (II). $0 \neq \delta_0$.

Choose $0 \leqslant u \in E$ with norm $\|u\| = 1$ and $\delta_0(u) \neq 0$. The set $X_u := \{\delta \in V(E) |\ \ \delta(u) = 1\}$ with $\sigma(E', E)$ is homeomorphic to the $Y_u = \{\delta \in V(E)_1 \mid \delta(u) \neq 0\}$ which is open in Max(E); the homeomorphism Φ: $Y_u \to X_u$ is given by Φ: $\delta \to \delta/\delta(u)$ [9, 2.1].

Let (δ_α) be a net in $V(E)_1$ weak*-converging to δ_0. Because of $\delta_0(u) > 0$ we may suppose $\delta_\alpha \in Y_u$ for all α. Then $\Phi(\delta_\alpha)$ converges to $\delta_0/\delta_0(u)$ in X_u (also weak*).

Let $\mathfrak{U}$ be a neighborhoodbase of $\delta_0/\delta_0(u)$ in X_u consisting of open subsets of X_u. Fix a downward directed net $(r_U)_{U \in \mathfrak{U}}$ of positive real numbers with infimum 0. Because of lim $\Phi(\delta_\alpha) = \delta_0/\delta_0(u)$, lim $\delta_\alpha(u) = \delta_0(u)$ and lim $\delta_\alpha(e) = \delta_0(e)$ for every $U \in \mathfrak{U}$ there is α_U such that (write $\delta_U := \delta_{\alpha_U}$)

$$\Phi(\delta_U) \in U, \qquad \delta_U(u) \in (\delta_0(u) - r_U,\ \delta_0(u) + r_U),$$
$$\delta_U(e) \in (\delta_0(e) - r_U,\ \delta_0(e) + r_U). \tag{3.10}$$

As $\|\delta_U\| = 1$ for every $0 < r < 1$ there is $f_{U,r} \in E$, $0 \leqslant f_{U,r}$ and $\|f_{U,r}\| \leqslant 1$, such that $1 - r < \delta_U(f_{U,r}) \leqslant 1$. Let $V_{U,r} := \{\delta \in X_u \mid |(\delta - \Phi(\delta_U))\ (f_{U,r})| < r\}$; the set $V_{U,r}$ is an open neighborhood of $\Phi(\delta_U)$ in X_U. The same is true for $W_{U,r} := U \cap V_{U,r}$.

Now, let Ψ: $X_u \to Y_u$ be the inverse of Φ; Ψ maps $\Phi(\delta_U)$ into δ_U and $W_{U,r}$ into a facially open neighborhood $\Psi(W_{U,r})$ of δ_U in Y_u. As Y_u is open in

$\mathrm{Max}(E)$ the set $\Psi(W_{U,r})$ is open, too. The complete regularity of $\mathrm{Max}(E)$ yields continuous real functions $\Phi_{U,r}$ on $\mathrm{Max}(E)$ such that

$$0 \leqslant \Phi_{U,r} \leqslant 1, \qquad \Phi_{U,r}(\delta_U) = 1, \qquad \Phi_{U,r}(\mathrm{Max}(E)\backslash\psi(W_{U,r})) = \{0\}, \tag{3.11}$$

which in turn define central operators $R_{U,r}$ with $0 \leqslant R_{U,r} \leqslant I$ and linked with the $\Phi_{U,r}$ by (3.6). Finally, define the operators $T_{U,r}\colon E \to E$ by

$$T_{U,r}(g) = \mu_0(g)\, R_{U,r}(f_{U,r}) + (I - R_{U,r})\,(g), \qquad g \in E.$$

The index set $\mathfrak{U} \times (0, 1)$ becomes upward directed by

$$(U, r) \leqslant (U', r') \qquad \text{iff} \quad U' \subset U \text{ and } r' \leqslant r.$$

Using the same method as in Case (I) it is easily seen that all operators $T_{U,r}$ are linear, positive and contractive. It remains to be shown that $\lim T_{U,r}(g) = g$ for all $g \in L$, but that not $\lim T_{U,r}(e) = e$.

Considering the easier part first, namely not $\lim T_{U,r}(e) = e$, evaluating $\delta_U(T_{U,r}(e))$ by means of (3.11) and (3.6) yields

$$\delta_U(T_{U,r}(e)) = \mu_0(e)\, \delta_U(f_{U,r}). \tag{3.12}$$

Because of $1 - r < \delta_U(f_{U,r}) \leqslant 1$ for all $U \in \mathfrak{U}$, $0 < r < 1$, (3.12) converges to $\mu_0(e)$ as $r \to 0$. On the other hand

$$\begin{aligned} |\delta_0(e) - \delta_U(T_{U,r}(e))| &\leqslant |\delta_U(e) - \delta_U(T_{U,r}(e))| + |\delta_0(e) - \delta_U(e)| \\ &\leqslant \|e - T_{U,r}(e)\| + r_U \end{aligned}$$

(use (3.10)), which implies (3.12) to converge to $\delta_0(e)$ if $(T_{U,r}(e))$ converges to e. As $\mu_0(e) \neq \delta_0(e)$ by assumption $(T_{U,r}(e))$ does not converge to e.

Finally, we turn to $\lim T_{U,r}(g)$ for $g \in L$. Fix $g \in L$ and $1 > \varepsilon > 0$. There is $U_1 \in \mathfrak{U}$ such that $\delta \in U_1$ implies

$$\left| \left(\delta - \frac{\delta_0}{\delta_0(u)} \right) (g) \right| < \frac{\varepsilon}{4}, \tag{3.13}$$

since g as a function on X_u is $\sigma(E', E)$-continuous and $\mathfrak{U}$ is a neighborhoods base of $\delta_0/\delta_0(u)$. Furthermore, by (3.10) $(\delta_U(u))$ converges to $\delta_0(u)$. Thus, there exists $U_2 \in \mathfrak{U}$ such that for all $U \in \mathfrak{U}$ with $U \subset U_2$ we have

$$\left| \frac{\delta_0(g)}{\delta_0(u)} \right| \; \left| 1 - \frac{\delta_0(u)}{\delta_U(u)} \right| < \frac{\varepsilon}{4}. \tag{3.14}$$

Again by the convergence of $(\delta_U(u))$ to $\delta_0(u) > 0$ there is $U_3 \in \mathfrak{U}$ such that $U \in \mathfrak{U}$ and $U \subset U_3$ imply

$$\delta_U(u) > \tfrac{1}{2}\,\delta_0(u). \tag{3.15}$$

Finally, let $U_0 \in \mathfrak{U}$ be contained in $U_1 \cap U_2 \cap U_3$, and

$$r_0 = \frac{\varepsilon}{4} \min\left(1, \frac{1}{|\delta_0(g)|}, \frac{\delta_0(u)}{2\,|\delta_0(g)|}\right). \tag{3.16}$$

Now, take $U \in \mathfrak{U}$ and $r \in (0, 1)$ such that $U \subset U_0$, $r \leqslant r_0$. For arbitrary $\delta \in V(E)_1$ we have to show

$$|\delta(T_{U,r}(g) - g)| < \varepsilon.$$

Using $\mu_0(g) = \delta_0(g)$ and (3.6) this becomes

$$\begin{aligned} |\delta(T_{U,r}(g) - g)| &= |\delta_0(g)\,\Phi_{U,r}(\delta)\,\delta(f_{U,r}) - \Phi_{U,r}(\delta)\,\delta(g)| \\ &= \varphi_{U,r}(\delta)\,|\delta_0(g)\,\delta(f_{U,r}) - \delta(g)|. \end{aligned} \tag{3.17}$$

By (3.11) $\Phi_{U,r}(\delta) = 0$ for $\delta \notin \Psi(W_{U,r})$; thus it suffices to consider the case $\delta \in \Psi(W_{U,r})$. But this implies $\Phi(\delta) \in W_{U,r} = U \cap V_{U,r}$, in particular

$$\left|\left(\frac{\delta}{\delta(u)} - \frac{\delta_U}{\delta_U(u)}\right)(f_{U,r})\right| < r.$$

Hence

$$\frac{\delta(f_{U,r})}{\delta(u)} \leqslant \frac{\delta_U(f_{U,r})}{\delta_U(u)} + r \leqslant \frac{1}{\delta_U(u)} + r$$

and

$$\frac{\delta_U(f_{U,r})}{\delta(u)} \geqslant \frac{\delta_U(f_{U,r})}{\delta_U(u)} - r \geqslant (1 - r)\,\frac{1}{\delta_U(u)} - r.$$

Assuming $\delta_0(g) \geqslant 0$ we therefore obtain

$$\begin{aligned} \delta_0(g)\,\delta(f_{U,r}) - \delta(g) &\leqslant \delta_0(g)\,\frac{\delta(u)}{\delta_U(u)} - \delta(g) + \delta_0(g)\,\delta(u)r \\ &\leqslant \delta(u)\left|\frac{\delta_0(g)}{\delta_U(u)} - \frac{\delta(g)}{\delta(u)}\right| + \delta_0(g)\,\delta(u)r \\ &\leqslant \left|\frac{\delta_0(g)}{\delta_U(u)} - \frac{\delta(g)}{\delta(u)}\right| + |\delta_0(g)|\,r \end{aligned}$$

and

$$\begin{aligned}\delta_0(g)\,\delta(f_{U,r})-\delta(g) &\geqslant (1-r)\,\delta_0(g)\frac{\delta(u)}{\delta_U(u)}-\delta(g)-\delta_0(g)\,\delta(u)r\\ &\geqslant \delta(u)\left[\frac{\delta_0(g)}{\delta_U(u)}-\frac{\delta(g)}{\delta(u)}\right]-r\delta_0(g)\delta(u)\left(1+\frac{1}{\delta_U(u)}\right)\\ &\geqslant -\left|\frac{\delta_0(g)}{\delta_U(u)}-\frac{\delta(g)}{\delta(u)}\right|-r\delta_0(g)\left(1+\frac{1}{\delta_U(u)}\right).\end{aligned}$$

Both inequalities together can be written as

$$\begin{aligned}|\delta_0(g)\,\delta(f_{U,r})-\delta(g)| &\leqslant \left|\frac{\delta_0(g)}{\delta_U(u)}-\frac{\delta(g)}{\delta(u)}\right|+r\,|\delta_0(g)|\left(1+\frac{1}{\delta_U(u)}\right)\\ &\leqslant \left|\left(\frac{\delta_0}{\delta_0(u)}-\frac{\delta}{\delta(u)}\right)(g)\right|+\left|\frac{\delta_0(g)}{\delta_0(u)}\right|\,\left|\frac{\delta_0(u)}{\delta_U(u)}-1\right|\\ &\quad +r\,|\delta_0(g)|+r\frac{|\delta_0(g)}{\delta_U(u)}.\end{aligned}$$

Now, because of $\Phi(\delta)=\delta/\delta(u)\in U\subset U_0\subset U_1$ (3.13) gives $|(\delta_0/\delta_0(u)-\delta/\delta(u))(g)|<\varepsilon/4$; also, because of (3.14) and $U\subset U_0\subset U_2$ $|\delta_0(g)/\delta_0(u)|\,|\delta_0(u)/\delta_U(u)-1|<\varepsilon/4$. Furthermore, $r\ |\delta_0(g)|\leqslant\varepsilon/4$ by (3.16) and $r\leqslant r_0$, and $r\ |\delta_0(g)|/\delta_U(u)\leqslant r_0\ (2\,|\delta_0(g)|/\delta_0(u)\leqslant\varepsilon/4$ by (3.15), (3.16) and $U\subset U_0\subset U_3$, $r\leqslant r_0$.

Altogether $|\delta_0(g)\ \delta(f_{U,r})-\delta(g)|<\varepsilon$, and by (3.17) and $\varphi_{U,r}(\delta)\leqslant 1$ we obtain our claim $|\delta(T_{U,r}(g)-g)|<\varepsilon$.

Finally, in the case $\delta_0(g)<0$ one easily shows the same estimate for $|\delta_0(g)\ \delta(f_{U,r})-\delta(g)|$ to hold true. Thus $|\delta(T_{U,r}(g)-g|<\varepsilon$ in any case and we are done.

To prove Theorem 2.11 we shall inspect the proof of Theorem 2.9 to show that in both Cases (I) and (II) considered there it is possible to index the operators $T_{K,f}$ and $T_{U,r}$ countably.

Proof of Theorem 2.11. First observe that a separable AM-space contains a topological order-unit (this even is true in the context of Banach lattices: [11, II.6.2]) and that therefore its structure space is completely regular [9, 2.11].

In Case (I) we had the operators indexed by $\mathscr{K}\times A$, where $\mathscr{K}$ was a fundamental system of compact subsets of Str(E), $A=\{f\in E\mid 0\leqslant f,\ \|f\|\leqslant 1\}$. Now, if E is separable, Str(E) is the union of a sequence $\mathscr{K}$ of compact subsets [9, 2.17]. That it can be chosen fundamental follows by

2.10 and 2.9 of the same paper. Furthermore A contains a dense sequence $\{f_n \mid n \in \mathbb{N}\}$.

Let $g_n := \bigvee_{i=1}^n f_i$ and $B = \{g_n \mid n \in \mathbb{N}\}$; thus with B replacing A and the fundamental sequence $\mathscr{K}$ of compact subsets of $\mathrm{Str}(E)$ the proof of Case (I) works as well but we have only a (double) sequence of operators to use. An appropriate diagonalization process then yields the assertion.

In Case (II) observe that we had used the neighborhoodbase $\mathfrak{U}$ of $\delta_0/\delta_0(u)$ in X_u only to establish (3.13). We might have taken also

$$\tilde{\mathfrak{U}} = \{U_{g,n}\}, \qquad U_{g,n} = \left\{\delta \in X_u \,\middle|\, \left|\left(\delta - \frac{\delta_0}{\delta_0(u)}\right)(g)\right| < \frac{1}{n}\right\}$$

with $g \in L$, $n \in \mathbb{N}$.

Now, if E is separable, L is too. Let $\{g_n \mid n \in \mathbb{N}\}$ be a dense sequence in L, $U_{mn} := U_{g_m,n}$ and $U_n = \bigcap_{i \leqslant n/j \leqslant n} U_{ij}$. Then $\mathfrak{U}' = \{U_n \mid n \in \mathbb{N}\}$ is monotonically decreasing and fundamental in $\{U_{mn} \mid m, n \in \mathbb{N}\}$. Then with the new index set $\mathfrak{U}' \times \{1/n \mid n \in \mathbb{N}\}$ for the operators constructed in Case (II) we obtain a double sequence $T_{mn} := T_{U_m}$, $1/n$ of positive linear contractions on E converging pointwise to the identity on the linear hull G of $\{g_n \mid n \in \mathbb{N}\}$, but not on e. An appropriate diagonalization process yields a sequence (T_n) which does the same. But since G is dense in L and the sequence (T_n) is norm bounded, we finally conclude that $\lim T_n(g) = g$ for all $g \in L$, but that $(T_n(e))$ does not converge to e.

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